Calculus II
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Test II

Instructions: Show complete work for full credit.

1. (*10 pts.*) Circle an answer to indicate whether the following statements are true or false:

(a) **T** **F** For all positive real numbers x and y, $\ln(x+y) = \ln(x) + \ln(y)$.

(b) **T** **F** If $x > 0$ then $\log_\pi(\sqrt{x}) = \dfrac{\ln x}{2 \ln \pi}$.

(c) **T** **F** The range of $f(x) = \sin^{-1}(2x)$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(d) **T** **F** The inverse of the function $f(x) = (\sqrt{2})^{4x}$ is $g(x) = \log_4(x)$.

(e) **T** **F** For all positive real numbers x and y, $\ln\left(\dfrac{x}{y}\right) = \ln(x) + \ln\left(\dfrac{1}{y}\right)$.

2. (*10 pts.*) Evaluate $\int \sin^7(x)\cos^3(x)\,dx$

Let $u = \sin x$
$du = \cos x\,dx$

$$I = \int \sin^7(x) \cdot \cos^2(x)\cos(x)\,dx$$

$$= \int \sin^7 x\,(1 - \sin^2 x)\cos x\,dx$$

$$= \int u^7(1-u^2)\,du = \frac{1}{8}u^8 - \frac{1}{10}u^{10} + C$$

$$= \frac{1}{8}\sin^8 x - \frac{1}{10}\sin^{10} x + C$$

3. (*10 pts.*) Evaluate $\int x^4 \log_2(3x)\,dx$

Let $u = \log_2 3x, \quad du = \frac{1}{3x \ln 2} \cdot 3dx$

Integrating by Parts

$dv = x^4 dx, \quad v = \frac{1}{5}x^5$

$$I = \frac{1}{5}x^5 \log_2 3x - \int \frac{1}{5} x^5 \cdot \frac{1}{x \cdot \ln 2}\,dx$$

$$= \frac{1}{5} x^5 \log_2 3x - \frac{1}{25 \ln 2} x^5 + C$$

4. (*10 pts.*) Evaluate $\int \frac{\pi^{\arctan(x)}}{1+x^2}\,dx$

Let $u = \arctan(x)$

$du = \frac{dx}{1+x^2}$

$$I = \int \pi^{\arctan x} \cdot \frac{dx}{1+x^2}$$

$$= \int \pi^u\,du$$

$$= \frac{\pi^u}{\ln \pi} + C = \frac{\pi^{\arctan x}}{\ln \pi} + C$$

5. (*20 pts.*) Calculate each of these limits, if they exist, and justify your calculation by indicating any indeterminate forms.

(a) $\lim_{x\to 0} \dfrac{7^x - 3^{2x}}{\sin(\pi x)}$ This is an indet. form of type $\frac{0}{0}$

$$L = \lim_{x\to 0} \frac{7^x - 3^{2x}}{\sin(\pi x)} \overset{\mathcal{L}}{=} \lim_{x\to 0} \frac{7^x \cdot \ln 7 - 2\cdot 3^{2x}\cdot \ln 3}{\pi \cos(\pi x)}$$

$$= \frac{7^0 \cdot \ln 7 - 2\cdot 3^0 \cdot \ln 3}{\pi}$$

$$= \underline{\underline{\frac{\ln 7 - 2\ln 3}{\pi}}}$$

(b) $\lim_{x\to\infty} \left(1 + \dfrac{2}{x}\right)^{3x}$ This is indet. form. of type 1^∞.

$$\ln L = \lim_{x\to\infty} \ln\left[\left(1+\frac{2}{x}\right)^{3x}\right] = \lim_{x\to\infty} \underbrace{3x\cdot \ln\left(1+\frac{2}{x}\right)}_{\text{indet. of type } \infty\cdot 0}$$

$$= \lim_{x\to\infty} \underbrace{\frac{3\ln\left(1+\frac{2}{x}\right)}{1/x}}_{\text{indet. form type } 0/0} \overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{\frac{3}{1+2/x}\cdot\left(\frac{-2}{x^2}\right)}{\left(-1/x^2\right)}$$

$$= \lim_{x\to\infty} \frac{6}{1+2/x} = 6, \quad \text{so } \ln L = 6, \text{ and}$$

$$\underline{\underline{\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x} = L = e^6}}$$

6. (*10 pts.*) Use the definition of inverse sine to show that $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1}x$, so $\sin y = x$. Differentiating,

$\cos y \cdot y' = 1$, OR $y' = \frac{1}{\cos y}$.

Using $\cos^2 x + \sin^2 x = 1$, we have

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\cos^2 y}}$$

choosing positive root b/c $\cos y \geq 0$ on domain $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

i.e. $$\boxed{\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}}$$

7. (*10 pts.*) Evaluate $\int \frac{x\,dx}{\sqrt{x^2-4x+1}}$

Notice

$x^2-4x+1 = (x-2)^2 - 4 + 1 = (x-2)^2 - 3$

$$I = \int \frac{x\,dx}{\sqrt{(x-2)^2-3}}$$

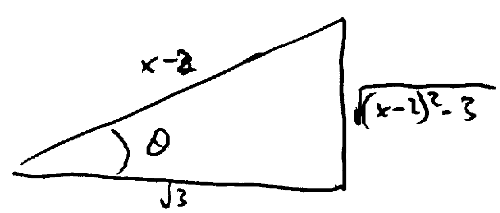
Let $x-2 = \sqrt{3}\sec\theta$, $dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

and $\sqrt{(x-2)^2-3} = \sqrt{3\sec^2\theta - 3} = \sqrt{3}\tan\theta$

$$I = \int \frac{(2+\sqrt{3}\sec\theta)(\sqrt{3}\sec\theta\tan\theta\,d\theta)}{\sqrt{3}\tan\theta} = \int \left(2\sec\theta + \sqrt{3}\sec^2\theta\right)d\theta$$

$$= 2\ln\left|\sec\theta + \tan\theta\right| + \sqrt{3}\tan\theta + C$$

$$= 2\ln\left|\frac{x-2}{\sqrt{3}} + \frac{\sqrt{(x-2)^2-3}}{\sqrt{3}}\right| + \sqrt{(x-2)^2-3} + C$$



$\sec\theta = \frac{x-2}{\sqrt{3}}$

$\tan\theta = \frac{\sqrt{(x-2)^2-3}}{\sqrt{3}}$