**Calculus II**
October 23rd, 2009

Name: Solutions

**Test II**

**Instructions:** Show complete work for full credit.

**1.** (*10 pts.*) Circle an answer to indicate whether the following statements are true or false:

(a) **T** **F** If $x > 0$ then $\log_5(5x) = \frac{\ln(x)}{\ln 5} + 1$. $= \log_5 x + \log_5 5 = \frac{\ln x}{\ln 5} + 1$

(b) **T** **F** For all positive real numbers $x$ and $y$, $\ln\left(\frac{x}{y}\right) = \ln(x) + \ln\left(\frac{1}{y}\right)$.

(c) **T** **F** For all positive real numbers $x$ and $y$, $\ln(x-y) = \frac{\ln(x)}{\ln(y)}$.

(d) **T** **F** The inverse of the function $f(x) = (\sqrt{3})^{4x}$ is $g(x) = \log_9(x)$.

(e) **T** **F** The domain of $f(x) = \tan^{-1}(2x)$ is $(-\infty, \infty)$.

**2.** (*10 pts.*) Evaluate $\int \sec^6(x)\,\tan^3(x)\,dx$

Let $u = \sec x$
$du = \sec x \tan x\, dx$

$$I = \int \sec^5 x \cdot \tan^2 x \cdot \sec x \tan x\, dx = \int \sec^5 x(\sec^2 x - 1)\sec x \tan x\, dx$$

$$= \int u^5(u^2-1)\,du = \int u^7 - u^5\,du$$

$$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{8}\sec^8 x - \frac{1}{6}\sec^6 x + C$$

**3.** (*10 pts.*) Evaluate $\int x^6 \log_3(2x)\,dx$

Let $u = \log_3(2x)$, $du = \frac{1}{2x \ln 3} \cdot 2d$

$dv = x^6$, $v = \frac{1}{7}x^7$

$$I = \frac{1}{7}x^7 \log_3 2x - \int \frac{1}{x \cdot \ln 3} \cdot \frac{1}{7}x^7\,dx$$

$$= \frac{1}{7}x^7 \log_3 2x \; \frac{1}{7\ln 3} \int x^6\,dx$$

$$= \frac{1}{7}x^7 \log_3 2x - \frac{1}{49 \ln 3} x^7 + C$$

**4.** (*10 pts.*) Evaluate $\int \frac{\pi^{\arcsin(x)}}{\sqrt{1-x^2}}\,dx$

Let $u = \arcsin(x)$

$du = \frac{1}{\sqrt{1-x^2}}dx$

Then

$$I = \int \pi^{\arcsin(x)} \cdot \frac{dx}{\sqrt{1-x^2}}$$

$$= \int \pi^u\,du = \frac{\pi^u}{\ln \pi} + C$$

$$= \frac{\pi^{\sin^{-1}(x)}}{\ln \pi} + C$$

**5.** (*20 pts.*) Calculate each of these limits, if they exist, and justify your calculation by indicating any indeterminate forms.

(a) $\underbrace{\lim_{x\to 0} \frac{\pi^{2x} - e^x}{\tan(\pi x)}}_{\text{indet. form type } \frac{0}{0}} \stackrel{\mathcal{L}}{=} \lim_{x\to 0} \frac{(2\ln\pi)\,\pi^{2x} - e^x}{\pi \sec^2(\pi x)}$

$= \frac{2\ln\pi - 1}{\pi}$ since $\sec(0) = +1$

(b) $\lim_{x\to\infty} \left(1 - \frac{3}{x}\right)^x$

This is indeterminate of form $1^\infty$

Let $L = \lim_{x\to\infty} \left(1-\frac{3}{x}\right)^x$

So

$\ln L = \lim_{x\to\infty} \ln\left(\left(1-\frac{3}{x}\right)^x\right) = \lim_{x\to\infty} \underbrace{x \cdot \ln\left(1-\frac{3}{x}\right)}_{\text{indet of type } \infty\cdot 0}$

$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{3}{x}\right)}{1/x}$ indet of type $\frac{0}{0}$

$\ln L \stackrel{\mathcal{L}}{=} \lim_{x\to\infty} \frac{\frac{1}{1-3/x}\cdot\left(\frac{3}{x^2}\right)}{\left(-1/x^2\right)} = \lim_{x\to\infty} \frac{-3}{1-3/x} = -3$

and so

$\lim_{x\to\infty} \left(1-\frac{3}{x}\right)^x = L = e^{-3} = \frac{1}{e^3}$

6. (*10 pts.*) Use the definition of inverse tangent to show that $\dfrac{d}{dx}\left(\tan^{-1}x\right) = \dfrac{1}{1+x^2}$.

Let $y = \tan^{-1}x$, so by def$\underline{\text{n}}$ $\tan y = x$.

differentiating implicitly, $\sec^2 y \cdot y' = 1$ so

$$y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$\underline{\text{Thus}}$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

7. (*10 pts.*) Evaluate $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+2x-2}}$

$$x^2+2x-2 = (x+1)^2-1-2 = \underline{(x+1)^2-3}$$

$$I = \int \frac{x\,dx}{\sqrt{(x+1)^2-3}}$$

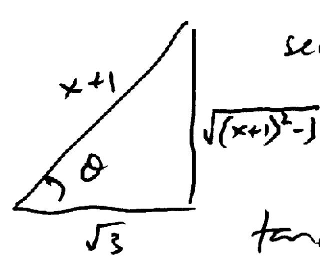
set $x+1 = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{(\sqrt{3}\sec\theta - 1)\sqrt{3}\sec\theta\tan\theta\,d\theta}{\sqrt{(\sqrt{3}\sec\theta)^2-3}} = \int \frac{(\sqrt{3}\sec\theta-1)\sqrt{3}\sec\theta\tan\theta\,d\theta}{\sqrt{3}\tan\theta}$$

$$= \int\left(\sqrt{3}\sec^2\theta - \sec\theta\right)d\theta = \underline{\sqrt{3}\tan\theta - \ln|\sec\theta+\tan\theta| + C}$$

$$\boxed{I = \sqrt{(x+1)^2-3} - \ln\left|\frac{x+1}{\sqrt{3}} + \frac{\sqrt{(x+1)^2-3}}{\sqrt{3}}\right| + C}$$




$\sec\theta = \dfrac{x+1}{\sqrt{3}}$

$\tan\theta = \dfrac{\sqrt{(x+1)^2-3}}{\sqrt{3}}$